Substitutions on compact infinite alphabets

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Primitive substitutions on finite alphabets generate a variety of mathematical models of quasicrystals. They exhibit geometric and statistical properties which are expected for such models; see [1] for a comprehensive treatment. These provide access to the investigation of the diffraction spectrum, as well as the spectrum of discrete Schrödinger operators whose potential functions depend on these aperiodic structures. The classical example is the Fibonacci substitution on the alphabet $\mathcal{A} = \{0,1\}$ given by $\varrho\colon 0 \to 01, 1 \to 0$. This gives rise to a point set whose diffraction comprises solely of Bragg peaks [1]. On the Schrödinger side, the resulting self-adjoint operator only admits purely singular continuous spectral measures, has a spectrum which is a Cantor set of zero Lebesgue measure, and exhibits anomalous transport [2]. In this talk, we will explore what lies beyond the finite alphabet regime and discuss parallelisms and surprises in the theory of substitutions on infinite (yet compact) alphabets. As a central example, we consider the (countable) alphabet $\mathcal{A} = \mathbb{N} \cup \{\infty\}$, given by $\varrho\colon 0 \to 01, n \to 0(n-1)(n+1)$ (for $0 < n < \infty$) and $\infty \to 0(\infty)(\infty)$; see [3,4,5,6]. We will demonstrate certain features that one cannot get from finite-alphabet substitutions [3,4], as well as generalizations of the example above to uncountable alphabets [4,5]. Time permitting, we will provide an outlook towards spectral theory.

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Periodicity of joint co-tiles

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An old theorem of Newman asserts that any tiling of $\mathbb{Z}$ by a finite set is periodic. A few years ago Bhattacharya proved the periodic tiling conjecture in $\mathbb{Z}^2$. Namely, he proved that for a finite subset F of $\mathbb{Z}^2$, if there exists a 'set of translations', $A \subseteq \mathbb{Z}^2$ such that $F \oplus A = \mathbb{Z}^2$ then there exists a periodic $A' \subseteq \mathbb{Z}^2$ such that $F \oplus A' = \mathbb{Z}^2$. The recent refutation of the periodic tiling conjecture in high dimensions due to Greenfeld and Tao motivates finding different generalizations of Newman's theorem and of Bhattacharya's theorem that hold in arbitrary dimension d. We formulate and prove such generalizations. We do so by studying the structure of joint co-tiles in $\mathbb{Z}^d$. In my talk I will discuss these generalizations. The talk is based on a joint work with Tom Meyerovitch and Shrey Sanadhya [1].

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Coverability of words and pictures

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A one-dimensional sequence w is said to be u-coverable if w is formed by overlapping or adjacent copies of its proper subword u. Given a finite word u, we study the properties of the shift space of all u-coverable words. Coverability notions on substitutions and their associated shift spaces are also developed and studied.

The concept of coverability can be extended naturally for higher-dimensional sequences. In two dimensions, coverability is a generalization of 2-periodicity but 1-periodicity and coverability are independent. Rectangular covers have been studied by Crochemore *et al.* [3] and Gamard and Richomme [4] for the finite and infinite cases, respectively.

In 1996, Gummelt [5] showed that Penrose rhombus tiling is coverable with a decorated decagon. Motivated by this, we investigate the coverability of two-dimensional pictures generated by taking the direct product two one-dimensional inflation tilings. We also consider how coverability is affected under variations of the product structure. Specifically, we determine the coverability of the direct product variations (DPV) of the Fibonacci inflation tiling studied in [1, 2].

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Patch frequencies in Penrose rhombic tilings

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We present an algorithm for an exact calculation of patch frequencies for the rhombic Penrose tiling [1]. The work is based on a construction of Penrose tilings via dualisation developed by Baake *et al.* [2]. By extending the known method for obtaining vertex configurations, we obtain the desired algorithm, which reduces the task to compute the area of an intersection of certain triangles. It is then used to determine the frequencies of several particular patches which appear in the literature. In particular, one can use it to estimate jumps at the integrated density of states of Laplacians on a graph induced by Penrose rhombic tilings, which were studied by Damanik *et al*. [3]. This method works for all tilings that can be obtained via the dualisation method, i.e. for example, for Ammann-Beenker tilings.

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Hexagonal metallic-mean tilings as aperiodic approximants of the honeycomb lattice

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Quasiperiodic system has been the subject of extensive research since the discovery of the Al-Mn quasicrystal [1]. Most quasicrystals are composed of clusters with a concentric shell structure that are arranged quasiperiodically. In some alloys, approximants with periodically arranged clusters have also been synthesized. Generally, the physical properties of approximants are similar to those of quasicrystals, making them useful for systematic studies. It is important to note that the periodic approximants approach the quasicrystal as the degree of approximation increases, which accelerates further investigations. Complementary treatments have also been discussed, where the quasiperiodic structure approaches the periodic one by changing the characteristic value, so-called, aperiodic approximants. Simple examples in two dimensions are the aperiodic approximants of the triangular and square lattices [2,3]. Since the hexagon covers the two-dimensional space, it is worth considering the aperiodic approximants of the honeycomb lattice.

We present aperiodic tilings composed of large and small hexagons and parallelograms. These tilings are characterized by the metallic-mean ratio $\tau_k = (k + \sqrt{k^2+4})/2$ with the natural number k. When $k = 1$, the tiling reduces to the three-tile golden-mean tiling recently discovered [4]. Therefore, our tilings are generalizations of the golden-mean tiling. As we increase k, the honeycomb domains bounded by the parallelograms grow in size, and the large hexagons occupy most of the two-dimensional space. This suggests that our tilings can be regarded as aperiodic approximants of the honeycomb lattice. By considering the substitution rules for the tiling, we obtain exact fractions of the eight types of vertices. We also observe sublattice imbalance in the vertices and discuss how this property relates to zero-energy states in the one-particle state for the tight-binding model. Additionally, we show that the spiky patterns in the lattice structure factors change as k increases. Finally, we address the perpendicular spaces for the tilings.

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